Area of the parabolic tangent triangle.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Let A_1,A_2,A_3 be three non-collinear point on parabola $4py = x^2, p > 0$ and let $B_1 = l_2 \cap l_3, B_2 = l_3 \cap l_1, B_3 = l_1 \cap l_2$ where l_1,l_2,l_3 are tangents to parabola at points A_1,A_2,A_3 respectively. Prove that ratio $\dfrac{[A_1,A_2,A_3]}{[B_1,B_2,B_3]}$ is constant and find it's value.

Solution.

Let $A_1(x_1,y_1),A_2(x_2,y_2),A_3(x_3,y_3)$ be three points on parabola $4py = x^2, p > 0$ in which, respectively, three tangents l_1,l_2,l_3 to parabola touches it, that is $l_i : 2p(y+y_i) = xx_i$ and $4py_i = x_i^2, i = 1,2,3.$

Since $\begin{cases} 2p(y+y_2) = xx_2 \\ 2p(y+y_3) = xx_3 \end{cases} \Leftrightarrow \begin{cases} 4p(y_3-y_2) = 2x(x_3-x_2) \\ y(x_3-x_2) = x_2y_3 - x_3y_2 \end{cases} \Leftrightarrow$

$$\begin{cases} x_3^2 - x_2^2 = 2x(x_3-x_2) \\ y(x_3-x_2) = \dfrac{x_2x_3^2 - x_3x_2^2}{4p} \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{x_2+x_3}{2} \\ y = \dfrac{x_2x_3}{4p} \end{cases}$$

then point B_1 of intersection tangents l_2 and l_3 have coordinates $\left(\dfrac{x_2+x_3}{2}, \dfrac{x_2x_3}{4p}\right)$.

Cyclic we obtain two another points $B_2\left(\dfrac{x_3+x_1}{2}, \dfrac{x_3x_1}{4p}\right)$, $B_3\left(\dfrac{x_1+x_2}{2}, \dfrac{x_1x_2}{4p}\right)$ of intersections $l_3 \cap l_1, l_1 \cap l_2$, respectively.

1.Area of Parabolic Triangle $\triangle A_1A_2A_3$ is:

$$[A_1A_2A_3] = \frac{1}{2}\left|\det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ \dfrac{x_1^2}{4p} & \dfrac{x_2^2}{4p} & \dfrac{x_3^2}{4p} \end{pmatrix}\right| = \frac{|(x_1-x_2)(x_2-x_3)(x_3-x_1)|}{8p}$$

2.Area of Tangent Triangle $\triangle B_1B_2B_3$ is:

$$[B_1,B_2,B_3] = \frac{1}{2}\left|\det \begin{pmatrix} 1 & 1 & 1 \\ \dfrac{x_2+x_3}{2} & \dfrac{x_3+x_1}{2} & \dfrac{x_1+x_2}{2} \\ \dfrac{x_2x_3}{4p} & \dfrac{x_3x_1}{4p} & \dfrac{x_1x_2}{4p} \end{pmatrix}\right| =$$

$$\frac{1}{16p}\left|\det \begin{pmatrix} 1 & 1 & 1 \\ x_2+x_3 & x_3+x_1 & x_1+x_2 \\ x_2x_3 & x_3x_1 & x_1x_2 \end{pmatrix}\right| = \frac{1}{16p}\left|\sum_{cyc}(x_1x_2(x_3+x_1) - x_3x_1(x_1+x_2))\right| =$$

$$\frac{1}{16p}\left|\sum_{cyc}(x_1^2x_2 - x_3x_1^2)\right| = \frac{||(x_1-x_2)(x_2-x_3)(x_3-x_1)||}{16p}.$$

Thus $\dfrac{[A_1A_2A_3]}{[B_1,B_2,B_3]} = 2.$